

CANONICAL HILBERT-BURCH MATRICES FOR IDEALS OF $k[x, y]$

ALDO CONCA, GIUSEPPE VALLA

ABSTRACT. An Artinian ideal I of $k[x, y]$ has many Hilbert-Burch matrices. We show that there is a canonical choice. As an application, we determine the dimension of certain affine Gröbner cells and their Betti strata recovering results of Ellingsrud and Strømme, Göttsche and Iarrobino.

1. INTRODUCTION

Let k be a field. Let R be the polynomial ring $k[x_1, \dots, x_n]$ and τ be a term order on R . Given a non-zero $f \in R$ we denote by $\text{Lt}_\tau(f)$ the largest term with respect to τ appearing in f . For an ideal I of R we denote by $\text{Lt}_\tau(I)$ the (monomial) ideal generated by $\text{Lt}_\tau(f)$ with $f \in I \setminus \{0\}$. Let E be a monomial ideal of R . Consider the set $V(E)$ of the homogeneous ideals I of R such that $\text{Lt}_\tau(I) = E$. The set $V(E)$ has a natural structure of affine variety. Namely, given I in $V(E)$, we can consider I as a point in an affine space \mathbf{A}^N with coordinates given by the coefficients of the non-leading terms in the reduced Gröbner basis of I , see Section 2 for details. The equations defining (at least set-theoretically) $V(E)$ can be obtained from Buchberger's Gröbner basis criterion. Provided $\dim_k R/E$ is finite, one can give the structure of affine variety also to the set $V_0(E)$ of the ideals I (homogeneous or not) such that $\text{Lt}_\tau(I) = E$.

These varieties play important roles in many contexts such as, for example, the study of various types of Hilbert schemes and the problem of deforming non-radical to radical or prime ideals, see [AS, Br, CRV, ES1, ES2, Go1, Go2, I1, I2, IY, MS].

Many of the equations defining $V(E)$ or $V_0(E)$ contain parameters that appear in degree 1 and that can be eliminated. It happens quite often that, after getting rid of the superfluous parameters, one is left with no equations, that is, the variety is an affine space. But, in general, $V(E)$ can be reducible and it can have irreducible components that are not affine spaces, see the examples 2.1, 2.2 and 2.3.

On the other hand, for $n = 2$ and $d = \dim_k R/E < \infty$, it is known that $V_0(E)$ and $V(E)$ are affine spaces. This is a consequence of general results of Bialynicki-Birula [BB1, BB2] on smooth varieties with k^* -actions. Here it is important to note that $V_0(E)$ coincides with the set of points of the Hilbert scheme $\text{Hilb}^d(\mathbf{A}^2)$ that degenerate to E under a suitable k^* -action associated to a weight vector representing the term order on monomials of degree $\leq d + 1$. By the analogy with Schubert cells for Grassmannians, we name $V_0(E)$ and $V(E)$ Gröbner cells.

Our goal here is to show that for $n = 2$ and τ the lexicographic order induced by $x > y$, both $V(E)$ and $V_0(E)$ can be described as affine spaces in a very explicit way, see 3.3. To achieve this goal we identify canonical Hilbert-Burch matrices of the ideals involved. The main point is to introduce (redundant) systems of generators

for the ideals in $V_0(E)$, that, instead of being themselves “simple”, have “simple” syzygies.

We can then easily deduce formulas for the dimensions of $V(E)$ and $V_0(E)$ and of two other subvarieties of $V_0(E)$, see 3.1. Dimension formulas for these varieties were originally obtained in [ES1, ES2, I1, Go2, IY]. In Section 4 we reprove and generalize some results of Iarrobino [I2] concerning the Betti strata of $V(E)$.

For standard facts on Gröbner bases we refer the reader to [KR] or [E]. The results of this paper were discovered, suggested and double-checked by extensive computer algebra experiments performed with CoCoA [Co].

2. $V(E)$ AS AN AFFINE VARIETY

With the notations introduced above, we first recall how $V(E)$ and $V_0(E)$ can be given the structure of affine varieties. For every minimal monomial generator m of E consider the polynomial

$$f_m = m - \sum \lambda(m, m') m'$$

where the sum is extended the monomials $m' \notin E$ such that $\deg m = \deg m'$ and $m' < m$ with respect to τ . Denote by N the total number of the parameters $\lambda(m, m')$. The property of being a Gröbner basis for the f_m 's is turned into the vanishing of polynomials, say B_1, \dots, B_r , on the parameters $\lambda(m, m')$. Since an ideal has a unique reduced Gröbner basis, the points of the affine variety of \mathbf{A}^N defined by the vanishing of the B_i are in bijection with the elements of $V(E)$. The polynomials B_i can be explicitly computed through the Buchberger's criterion for Gröbner basis. There are many degrees of freedom in the application of the Buchberger's criterion (e.g. one can use all the S-pairs or carefully choosen subsets of them, the reduction process can be performed in various ways, and so on). So the actual nature of the polynomials B_i depend on these choices but, of course, not the variety that they define.

Similarly, if $\dim_k R/E$ is finite, one can give the structure of affine variety to $V_0(E)$ by dropping the assumption that $\deg m = \deg m'$ in the definition of f_m .

As said in the Introduction, the varieties $V(E)$ and $V_0(E)$ quite often are affine spaces. Roughly speaking, what happens is the following. Say m, n are monomial generators of E , $m' < m$ and $t = m'n/\text{GCD}(m, n)$ satisfies $t \notin E$. Then the coefficient of t in the S -polynomial associated to f_m and f_n is just $\lambda(m, m')$ or $\lambda(m, m') - \lambda(n, n')$ depending on whether there exists $n' < n$ such that $t = n'm'/\text{GCD}(m, n)$. Performing the reduction procedure, $\lambda(m, m')$ cannot be cancelled because at each iteration the degree of the coefficients involved increases by 1. At the end of the reduction procedure, the coefficient of t in the polynomial we are left with must vanish. Therefore we have equations of the form:

$$(2.1) \quad \lambda(m, m') + B = 0 \quad \text{or} \quad \lambda(m, m') - \lambda(n, n') + B = 0$$

where B is a polynomial in the $\lambda(*, *)$ not involving monomials of degree 1. Of course if B does not involve $\lambda(m, m')$ at all then we can use 2.1 to get rid of the parameter $\lambda(m, m')$ from the equations. This elimination process can be iterated. In many cases, at the end of the elimination process, the equations vanish completely, and this shows that the associated variety is an affine space. We have implemented this rough algorithm in CoCoA [Co]. We have tested, for instance, that for $\tau = \text{Lex}$,

$n = 3$ and E any ideal generated by monomials of degree 3 then $V(E)$ is an affine space.

The following examples show that in general the variety $V(E)$ has a more complicated structure. For simplicity, the coordinates of the ambient affine spaces $\lambda(m, m')$ are denoted by a_i .

Example 2.1. Set $n = 3$, $E = (x_3^4, x_2^4, x_1x_2^2x_3, x_1^3x_3)$ and $\tau = \text{Lex}$. Then $V(E)$ is a subvariety of \mathbf{A}^{17} , the inclusion being given by the parametrization:

$$\begin{array}{lcl} x_1^3x_3 & -x_1^2x_2^2a_1 - x_1^2x_2x_3a_2 - x_1^2x_3^2a_3 - x_1x_2^3a_4 - x_1x_2x_3^2a_5 - x_1x_3^3a_6 \\ & -x_2^3x_3a_7 - x_2^2x_3^2a_8 - x_2x_3^3a_9, \\ x_1x_2^2x_3 & -x_1x_2x_3^2a_{10} - x_1x_3^3a_{11} - x_2^3x_3a_{12} - x_2^2x_3^2a_{13} - x_2x_3^3a_{14}, \\ x_2^4 & -x_2^3x_3a_{15} - x_2^2x_3^2a_{16} - x_2x_3^3a_{17}, \\ x_3^4 & \end{array}$$

Buchberger's criterion gives 3 equations. Two of them can be written as:

$$\begin{aligned} a_{14} &= -a_{10}^2a_{12} - a_{10}a_{13} - a_{11}a_{12}, \\ a_9 &= 2a_1a_{10}^2a_{12}a_{15} + \text{other 46 terms in the } a_i\text{'s not involving } a_9 \text{ and } a_{14}. \end{aligned}$$

Setting $b_{17} = a_{10}^3 - a_{10}^2a_{15} + 2a_{10}a_{11} - a_{10}a_{16} - a_{11}a_{15} - a_{17}$, the third equation is $a_1b_{17} = 0$. Hence $V(E)$ has two irreducible components both isomorphic to \mathbf{A}^{14} .

Example 2.2. Let $n = 4$, $E = (x_4^2, x_2x_4, x_2^2, x_1x_4)$ and $\tau = \text{Lex}$. Then $V(E)$ is a subvariety of \mathbf{A}^8 , the inclusion being given by the parametrization:

$$\begin{array}{lcl} x_4^2, & & \\ x_2x_4 & -x_3^2a_1 - x_3x_4a_2, & \\ x_2^2 & -x_2x_3a_3 - x_3^2a_4 - x_3x_4a_5, & \\ x_1x_4 & -x_2x_3a_6 - x_3^2a_7 - x_3x_4a_8. & \end{array}$$

The parameters a_1, a_7, a_4 can be eliminated, so that $V(E)$ is indeed contained in \mathbf{A}^5 . After renaming $b_3 = 2a_2 - a_3$ the defining ideal of $V(E)$ in \mathbf{A}^5 takes the form b_3a_6, a_5a_6 . Hence $V(E)$ has two irreducible components, one isomorphic to \mathbf{A}^3 and the other to \mathbf{A}^4 .

Example 2.3. Let $n = 4$, $E = (x_4^2, x_2x_4, x_1x_4, x_1x_2, x_1^2)$ and $\tau = \text{Lex}$. Then $V(E)$ is a subvariety of \mathbf{A}^{16} , the inclusion being given by the parametrization:

$$\begin{array}{lcl} x_4^2, & & \\ x_2x_4 & -x_3^2a_1 - x_3x_4a_2, & \\ x_1x_4 & -x_2^2a_3 - x_2x_3a_4 - x_3^2a_5 - x_3x_4a_6, & \\ x_1x_2 & -x_1x_3a_7 - x_2^2a_8 - x_2x_3a_9 - x_3^2a_{10} - x_3x_4a_{11}, & \\ x_1^2 & -x_1x_3a_{12} - x_2^2a_{13} - x_2x_3a_{14} - x_3^2a_{15} - x_3x_4a_{16}. & \end{array}$$

The parameters

$$a_1, a_3, a_4, a_5, a_{13}, a_{10}, a_{14}, a_{15}$$

can be eliminated, so that $V(E)$ is indeed contained in \mathbf{A}^8 . After renaming

$$b_9 = a_2a_8 + a_7a_8 - a_6 + a_9, \quad b_{12} = 2a_6 + b_9 - a_{12}, \quad b_7 = a_2 - a_7$$

the defining ideal of $V(E)$ in \mathbf{A}^8 takes the form

$$(b_7b_9, b_9b_{12}, a_{11}b_{12} - b_7a_{16}).$$

Hence $V(E)$ has two components, one is isomorphic to \mathbf{A}^6 and the other is a quadric hypersurface of rank 4 in \mathbf{A}^7 .

3. IDEALS IN $k[x, y]$

Form now on, let k be a field, $R = k[x, y]$ be the polynomial ring over k . We equip R with the lexicographic term order $>$ induced by $x > y$.

Given a monomial ideal $E \subset R$ with $\dim_k R/E < \infty$ we want to describe the set of ideals:

$$V_0(E) = \{I \text{ such that } \text{Lt}(I) = E\}$$

and its subsets

$$V_1(E) = \{I \text{ such that } \text{Lt}(I) = E \text{ and } y \in \sqrt{I}\},$$

$$V_2(E) = \{I \text{ such that } \text{Lt}(I) = E \text{ and } \sqrt{I} = (x, y)\},$$

$$V(E) = V_3(E) = \{I \text{ such that } \text{Lt}(I) = E \text{ and } I \text{ is homogeneous}\}.$$

Our goal is to prove Theorem 3.3. As a corollary we have:

Corollary 3.1. *The set $V_0(E)$ is an affine space. The subsets $V_1(E)$, $V_2(E)$ and $V_3(E)$ are also affine spaces, indeed coordinate subspaces of $V_0(E)$. Furthermore*

$$\dim V_i(E) = \begin{cases} \dim_k R/E + \min\{j : y^j \in E\} & \text{if } i = 0, \\ \dim_k R/E & \text{if } i = 1, \\ \dim_k R/E - \min\{j : x^j \in E\} & \text{if } i = 2, \\ \#\mathcal{S}(E) & \text{if } i = 3. \end{cases}$$

where $\mathcal{S}(E)$ is a set described below and $\#\mathcal{S}(E)$ denotes its cardinality.

The dimension formulas for $V_0(E)$, $V_1(E)$ and $V_2(E)$ have been proved originally in [ES1, ES2]. A dimension formula for $V_3(E)$ appears in [I1, IY] for lex-segments E and in [Go1] for general E .

To prove 3.1 one could try to analyze the equations coming from Buchberger's criterion. But this turns out to be quite difficult. Instead we parametrize the syzygies and identify canonical Hilbert-Burch matrices.

We introduce a piece of notation. Given a monomial ideal E such that $\dim_k R/E$ is finite, we set $t = \min\{j : x^j \in E\}$, $m_0 = 0$, and for every $1 \leq i \leq t$ $m_i = \min\{j : x^{t-i}y^j \in E\}$. It is clear that $m_0 = 0 < m_1 \leq m_2 \leq \dots \leq m_t$,

$$E = (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t})$$

and $\dim_k R/E = \sum_{i=0}^t m_i$. These generators of E are not minimal in general. They minimally generate E if and only if $m_0 < m_1 < m_2 < \dots < m_t$, that is, E is a lex-segment ideal. By construction, the correspondence

$$E \leftrightarrow (m_0, \dots, m_t)$$

is a bijection between monomial ideals of R with radical equal to (x, y) and sequences of integers $0 = m_0 < m_1 \leq m_2 \leq \dots \leq m_t$.

Given E or, equivalently (m_0, \dots, m_t) , we set

$$d_i = m_i - m_{i-1}$$

for $i = 1, \dots, t$. Here $d_1 > 0$ and $d_i \geq 0$ for every $i = 2, \dots, t$. Clearly, E can be as well described via the vector (d_1, \dots, d_t) . Furthermore, the lex-segment correspond exactly to the vectors with $d_i > 0$ for $i = 1, \dots, t$.

The matrix

$$M_0(E) = \begin{pmatrix} y^{d_1} & 0 & 0 & \cdots & 0 & 0 \\ -x & y^{d_2} & 0 & \cdots & 0 & 0 \\ 0 & -x & y^{d_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -x & y^{d_{t-1}} & 0 \\ 0 & 0 & 0 & 0 & -x & y^{d_t} \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$$

has size $(t+1) \times t$ and is a Hilbert-Burch matrix of E in the sense that the (signed) t -minors of $M_0(E)$ are the monomials $x^{t-i}y^{m_i}$ and the columns generate their syzygy module.

The matrix $M_0(E)$ represents a map from $F_1 = \oplus_{i=1}^t R(-t+i-m_i-1)$ to $F_0 = \oplus_{i=1}^{t+1} R(-t+i-1-m_{i-1})$. It is useful to consider also the corresponding degree matrix $U(E) = (u_{ij})$. The entries of $U(E)$ are the degrees of the (homogeneous) entries of every matrix representing a map of degree 0 from F_1 to F_0 . We have

$$(3.1) \quad u_{ij} = m_j - m_{i-1} + i - j \text{ for } i = 1, \dots, t+1 \text{ and } j = 1, \dots, t$$

Notice that $u_{ii} = m_i - m_{i-1} = d_i$ and $u_{i+1,i} = 1$ for every $i = 1, \dots, t$. Define:

$$\mathcal{S}(E) = \{(i, j) : 1 \leq j < i \leq t+1 \text{ and } 0 \leq u_{ij} < d_j\}$$

Definition 3.2. Let $T_0(E)$ be the set of $(t+1) \times t$ matrices $N = (n_{i,j})$ where

$$n_{i,j} = \begin{cases} 0 & \text{if } i < j \\ \text{a polynomial in } k[y] \text{ of degree } < d_j & \text{if } i \geq j \end{cases}$$

Further consider the following conditions:

- (1) $n_{i,i} = 0$ for every $i = 1, \dots, t$.
- (2) For every j such that $d_j > 0$ the polynomial $n_{i,j}$ has no constant term for every $i = j+1, \dots, k+1$ where $k = \max\{v : j \leq v \leq t \text{ and } m_v = m_j\}$.
- (3)

$$n_{i,j} = \begin{cases} 0 & \text{if } (i, j) \notin \mathcal{S}(E) \\ p_{ij}y^{u_{ij}} & \text{if } (i, j) \in \mathcal{S}(E) \end{cases}$$

with $p_{ij} \in k$. Accordingly we define

$$T_1(E) = \{N \in T_0(E) : N \text{ satisfies (1)}\}$$

$$T_2(E) = \{N \in T_0(E) : N \text{ satisfies (1) and (2)}\}$$

$$T_3(E) = \{N \in T_0(E) : N \text{ satisfies (3)}\}$$

Theorem 3.3. *For every monomial ideal E the map $\phi : T_0(E) \rightarrow V_0(E)$ defined by sending $N \in T_0(E)$ to the ideal of t -minors of the matrix $M_0(E) + N$ is a bijection. Furthermore, the restriction of ϕ induces bijections between $T_i(E)$ and $V_i(E)$ for $i = 1, 2, 3$.*

By construction, the sets $T_i(E)$ are affine spaces and their dimension can be easily computed from their defining conditions. Therefore Theorem 3.1 is an immediate consequence of 3.3.

Before embarking in the proof of 3.3 let us consider one example.

Example 3.4. Let $E = (x^3, xy^3, y^5) = (x^3, x^2y^3, xy^3, y^5)$. Then $m = (0, 3, 3, 5)$, $d = (3, 0, 2)$ and

$$M_0(E) = \begin{pmatrix} y^3 & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x \end{pmatrix} \quad U(E) = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

We have $t = 3$, $\min\{i : y^i \in E\} = 5$, $\dim R/E = 11$ and $\#\mathcal{S}(E) = 4$.

The matrices in $T_0(E)$ have the form:

$$\begin{pmatrix} n_{1,1} & 0 & 0 \\ n_{2,1} & 0 & 0 \\ n_{3,1} & 0 & n_{3,3} \\ n_{4,1} & 0 & n_{4,3} \end{pmatrix}$$

where the $n_{i,1}$ are polynomials in y of degree < 3 and the $n_{i,3}$ are polynomials in y of degree < 2 . The matrices in $T_1(E)$ are those of $T_0(E)$ such that $n_{1,1} = 0$ and $n_{3,3} = 0$. The matrices in $T_2(E)$ are those of $T_1(E)$ such that $n_{2,1}, n_{3,1}, n_{4,3}$ have no constant term. Finally the matrices in $T_3(E)$ have the form:

$$\begin{pmatrix} 0 & 0 & 0 \\ p_{21}y & 0 & 0 \\ p_{31}y^2 & 0 & 0 \\ p_{41}y & 0 & p_{43}y \end{pmatrix}$$

where the $p_{ij} \in k$.

As predicted by 3.1 we get $\dim T_0(E) = 16$, $\dim T_1(E) = 11$, $\dim T_2(E) = 8$, and $\dim T_3(E) = 4$.

The proof of 3.3 consists of the following steps:

(Step 1) The map ϕ is well-defined.

(Step 2) The map ϕ is bijective.

(Step 3) For $i = 1, 2, 3$ we have $\phi(N) \in V_i(E)$ iff $N \in T_i(E)$.

Let us begin with

Proof of Step 1. For $N \in T_0(E)$ set $I = \phi(N)$. We show that $\text{Lt}(I) = E$. For $i = 0, \dots, t$ let f_i be $(-1)^{t-i}$ times the determinant of the submatrix of $M_0(E) + N$ obtained by deleting the $(i+1)$ -th row. By construction, $\text{Lt}(f_i) = x^{t-i}y^{m_i}$ and $\text{Lc}(f_i) = 1$. We show that f_0, \dots, f_t form a Gröbner basis of I . The syzygy module of leading terms of the f_i is generated by the syzygies

$$(3.2) \quad y^{d_i}(x^{t-i+1}y^{m_{i-1}}) - x(x^{t-i}y^{m_i}) = 0$$

with $i = 1, \dots, t$. To prove that the f_i 's form a Gröbner basis it is enough to show that the S-polynomials associated to these syzygies reduce to 0. Since we have

$$y^{d_i}f_{i-1} - xf_i + \sum_{j=i-1}^t n_{j+1,i}f_j = 0$$

it is enough to show that if $y^{d_i}f_{i-1} - xf_i \neq 0$ then $\text{Lt}(n_{j+1,i}f_j) \leq \text{Lt}(y^{d_i}f_{i-1} - xf_i)$ for every $n_{j+1,i} \neq 0$. Note that the non-zero factors $n_{j+1,i}f_j$ have leading terms involving different powers of x . Hence $\max(\text{Lt}(n_{j+1,i}f_j) : n_{j+1,i} \neq 0) = \text{Lt}(y^{d_i}f_{i-1} - xf_i)$. \square

Step 2 will be a corollary of the following two lemmas:

Lemma 3.5. *Let I be an ideal of R such that $\text{Lt}(I) = E$ and let $f_0, \dots, f_t \in I$ such that $\text{Lt}(f_i) = x^{t-i}y^{m_i}$ and $\text{Lc}(f_i) = 1$. Then for every $f \in I$ such that $\text{Lt}(f) = x^{t-i}y^b$ for some $0 \leq i \leq t$ there exist polynomials $g_j \in k[y]$ with $j = i, \dots, t$ with $\deg g_i = b - m_i$ such that $f + g_i f_i + \dots + g_t f_t = 0$.*

Proof. By assumption, f_0, \dots, f_t is a Gröbner basis of I . Hence $x^{t-i}y^b$ is divisible by some $x^{t-j}y^{m_j}$. Hence $t - j \leq t - i$ and $m_j \leq b$. It follows that $i \leq j$ and $m_i \leq m_j \leq b$. Therefore $f - \text{Lc}(f)y^{b-m_i}f_i$ is still in I and has a smaller leading term (if it is non-zero). We get the desired representation by iterating the procedure. \square

Lemma 3.6. *Let I be an ideal of R such that $\text{Lt}(I) = E$. Then there exist $f_0, \dots, f_t \in I$ such that:*

- (1) $\text{Lt}(f_i) = x^{t-i}y^{m_i}$ and $\text{Lc}(f_i) = 1$ for every $i = 0, \dots, t$.
- (2) For every $i = 1, \dots, t$ there exists $n_{j+1,i} \in k[y]$ with $i - 1 \leq j \leq t$ and $\deg n_{j+1,i} < d_i$ such that

$$(3.3) \quad y^{d_i} f_{i-1} - x f_i + \sum_{j=i-1}^t n_{j+1,i} f_j = 0$$

Furthermore the polynomials f_i and $n_{j+1,i}$ with these properties are uniquely determined by I .

Proof. We prove the existence first. A set of polynomials $f_0, \dots, f_t \in I$ satisfying (1) clearly exists. We show how to modify them in order to fulfill (2). For a given k , $1 \leq k \leq t$ suppose that we have already modified f_k, \dots, f_t so that (1) is still fulfilled and that (2) is fulfilled for $i = k + 1, \dots, t$. We show how to modify f_{k-1} in order to fulfill (2) for $i = k$. Note that $y^{d_k} f_{k-1} - x f_k$ is in I and involves only terms with x -exponent $\leq t - (k - 1)$ and that if $x^{t-(k-1)}y^b$ is indeed present then $b < m_k$. By 3.5 we have that there exists $g_{k-1}, \dots, g_t \in k[y]$ such that g_{k-1} is either 0 or of degree $< d_k$ and

$$(3.4) \quad y^{d_k} f_{k-1} - x f_k + g_{k-1} f_{k-1} + g_k f_k + \dots + g_t f_t = 0$$

Set $h = y^{d_k} + g_{k-1}$ and perform for $j = k, \dots, t$ division with remainder: $g_j = h q_j + r_j$ with $q_j, r_j \in k[y]$ and r_j either 0 or of degree $< d_k$. Then we have

$$(3.5) \quad y^{d_k} f'_{k-1} - x f_k + g_{k-1} f'_{k-1} + r_k f_k + \dots + r_t f_t = 0$$

with $f'_{k-1} = f_{k-1} + q_k f_k + \dots + q_t f_t$. Note that f'_{k-1} is in I and $\text{Lt}(f'_{k-1}) = \text{Lt}(f_{k-1})$ and $\text{Lc}(f'_{k-1}) = \text{Lc}(f_{k-1})$. We may replace f_{i-1} with f'_{i-1} and 3.5 is the desired relation.

We prove now the uniqueness of the f_i 's and $n_{j+1,i}$ fulfilling (1) and (2). Suppose we have other polynomials f'_i and $n'_{j+1,i}$'s fulfilling (1) and (2). Note that $f_t = f'_t$ since they are both the monic generator of $I \cap k[y]$. So we may assume that $f_j = f'_j$ for $j = k, \dots, t$ and show that $f_{k-1} = f'_{k-1}$. By assumption we have equations:

$$(3.6) \quad y^{d_k} f_{k-1} - x f_k + n_{k,k} f_{k-1} + \sum_{j=k}^t n_{j+1,k} f_j = 0$$

$$(3.7) \quad y^{d_k} f'_{k-1} - x f'_k + n'_{k,k} f'_{k-1} + \sum_{j=k}^t n'_{j+1,k} f'_j = 0$$

where the $n_{j+1,k}$ and $n'_{j+1,k}$ are polynomials in $k[y]$ of degree $< d_k$.

By 3.5 applied with $f = f'_{k-1}$ we have an equation:

$$(3.8) \quad f'_{k-1} = f_{k-1} + g_k f_k + \cdots + g_t f_t$$

with $g_j \in k[y]$. Set $h = y^{d_i} + n_{k,k}$ and $h' = y^{d_i} + n'_{k,k}$. Replacing f'_{k-1} in 3.7 with the right hand side of 3.8 and then subtrating 3.6 we obtain:

$$(3.9) \quad (h' - h) f_{k-1} + \sum_{j=k}^t (h' g_j + n'_{k,j+1} - n_{k,j+1}) f_j = 0$$

Since the leading terms of the f_i 's involves distinct powers of x , the f_i 's are linearly independent over $k[y]$. Hence the coefficients $h' g_j + n'_{k,j+1} - n_{k,j+1}$ of 3.9 must be 0. Therefore $h' g_j = -n'_{k,j+1} + n_{k,j+1}$. But $n'_{k,j+1} + n_{k,j+1}$ has degree $< d_k$ and h' has degree d_k . Therefore $g_j = 0$ for every j and hence $f_{k-1} = f'_{k-1}$. Having shown that the f_i 's fulfilling (1) and (2) are uniquely determined by I , it remains that the coefficients $n_{j+1,i}$ are also uniquely determined. This is easy: given others coefficients $n'_{j+1,i}$ satisfying (2), say

$$(3.10) \quad y^{d_i} f_{i-1} - x f_i + \sum_{j=i-1}^t n'_{j+1,i} f_j = 0$$

we may subtract 3.3 from 3.10 and get

$$\sum_{j=i-1}^t (n'_{j+1,i} - n_{j+1,i}) f_j = 0.$$

This implies $n'_{j+1,i} = n_{j+1,i}$ by the linear indipendence of the f_i 's over $k[y]$. \square

We are ready to prove:

Proof of Step 2. We first prove that ϕ is injective. Suppose $I = \phi(N) = \phi(N')$ for matrices $N, N' \in T_0(E)$. We have seen in the proof of Step 1 that the signed t -minors f_0, \dots, f_t of $M_0(E) + N$ fulfill (1) and (2) of 3.6. The same it is true for the signed t -minors f'_0, \dots, f'_t of $M_0(E) + N'$. By the uniqueness of the f_i 's in 3.6 we have that $f_i = f'_i$ for every i . By the uniqueness of the coefficients of the equation of 3.6 the conclude that $N = N'$.

We show now that ϕ is surjective. Let $I \in V_0(E)$. We may find $f_0, \dots, f_t \in I$ satisfying (1) and (2) of 3.6. The equation 3.3 is the reduction to 0 of the S -polynomial corresponding to the syzygy 3.2 among the leading terms. As we know that these syzygies generate the syzygy module of the leading term of the f_i , Schreyer's theorem implies that the equations 3.3 give a system of generators for the syzygy module of the f_i 's. The corresponding matrix is of the form $M_0(E) + N$ with $N \in T_0(E)$ and the Hilbert-Burch theorem implies that $\phi(N) = I$. \square

Now we prove:

Proof of Step 3. Throughout the proof, N denotes a matrix in $T_0(E)$, $I = \phi(N)$ and f_0, \dots, f_t the signed t -minors of $M_0(E) + N$.

Since the f'_i 's form a Gröbner basis with respect to the lexicographic order, then $f_t = \prod_{i=1}^t (y^{d_i} + n_{i,i})$ generates $I \cap k[y]$. We have that $y \in \sqrt{I}$ iff f_t divides some power of y . But this is clearly equivalent to the vanishing of $n_{i,i}$ for $i = 1, \dots, t$. This proves that $N \in T_1(E)$ iff $\phi(N) \in V_1(E)$.

To prove that $N \in T_2(E)$ iff $\phi(N) \in V_2(E)$ we may assume that $N \in T_1(E)$ and we show that $\sqrt{I} = (x, y)$ iff the N fulfills condition (2) of 3.2. As we know already that $y \in \sqrt{I}$, we have that $\sqrt{I} = \sqrt{I + (y)}$. Replace y with 0 in $M_0(E) + N$, and call W_1 the resulting matrix. The first row of W_1 is 0 (since $d_1 > 0$). Denote by W the submatrix of W_1 obtained by deleting the first row. By construction $I + (y) = (\det W, y)$. We have to show that $\det W$ is a power of x iff N fulfills condition (2) of 3.2.

Let $C = \{i : i = 1, \dots, t \text{ and } d_i > 0\}$, say $C = \{i_1, \dots, i_p\}$ with $i_1 < \dots < i_p$. By assumption, $i_1 = 1$ and we set $i_{p+1} = t + 1$ by convention. The matrix W has a block decomposition

$$W = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ * & J_2 & 0 & \cdots & 0 \\ * & * & J_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & * & J_p \end{pmatrix}$$

where each J_v is a square block of size, say, $u = i_{v+1} - i_v$ and has the form

$$\begin{pmatrix} -x + a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_2 & -x & 1 & 0 & \cdots & 0 \\ a_3 & 0 & -x & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & 0 & \cdots & 0 & -x & 1 \\ a_u & 0 & \cdots & 0 & 0 & -x \end{pmatrix}$$

where $a_j = n_{i_v+j, i_v}(0)$ for $j = 1, \dots, u$. Now $\det W = \prod_v \det J_v$. The determinant of the matrix J_v is, up to sign, $x^u - a_1 x^{u-1} - a_2 x^{u-2} - \dots - a_u$. Hence $\det W$ is a power of x if and only if the coefficients a_j in every the J_v are 0. This is condition (2) of 3.2.

Finally we have to show that $N \in T_3(E)$ iff I is homogeneous. The “only if” direction is an immediate consequence of the fact that the matrix $M_0(E) + N$ is homogeneous for every $N \in T_3(N)$. The “if” direction follows from the observation that the polynomials f_i and $n_{i,j}$ of 3.6 are homogeneous if we start with a homogeneous ideal I . \square

4. BETTI STRATA OF $V(E)$

Let $h = h(z)$ be the Hilbert series of a graded Artinian quotient of R . It is known that $h(z)$ is of the form

$$h(z) = 1 + 2z + \cdots + cz^{c-1} + \sum_{j=c}^s h_j z^j$$

with $s + 1 \geq c \geq h_c \geq \dots \geq h_s > 0$. Denote by $\mathbb{G}(h)$ the variety that parametrizes graded ideals I in R such that the Hilbert series $h_{R/I}(z) = h(z)$. Iarrobino proved in [I1] that $\mathbb{G}(h)$ is a smooth projective variety whose dimension is given by the beautiful formula:

$$(4.1) \quad \dim \mathbb{G}(h) = h_c + \sum_{j=c}^s p_j p_{j+1}$$

where $p(z) = \sum_0^{s+1} p_i z^i = (1-z)h(z)$ is the first difference of $h(z)$.

Among the ideals with Hilbert series $h(z)$, the lex-segment plays a special role. We denote it by $L(h)$ or just L if $h(z)$ is clear from the context. If $\text{char } k = 0$, then $V(L)$ is dense in $\mathbb{G}(h)$ so that $\dim V(L) = \dim \mathbb{G}(h)$. Therefore, according to Corollary 3.1, we have

$$(4.2) \quad \dim \mathbb{G}(h) = \#\mathcal{S}(L)$$

To double-check, the suspicious reader can show directly that the right-hand side of the formulas 4.1 and 4.2 indeed coincide. It is a simple, but not obvious, exercise.

We come now to study the Betti strata of $V(E)$. For a homogeneous ideal I in $k[x, y]$ denote by $\beta_{i,j}(I)$ the (i, j) -th Betti number. In particular, $\beta_{0,j}(I)$ is the number of minimal generators of I of degree j . It is well known, each pairs of the three sets of invariants $\{\beta_{0,j}(I)\}_j$, $\{\beta_{1,j}(I)\}_j$ and the $\{\dim I_j\}_j$ determine the third. Given integers j and u we define:

$$V(E, j, u) = \{I \in V(E) : \beta_{0,j}(I) = u\}$$

$$V(E, j, \geq u) = \{I \in V(E) : \beta_{0,j}(I) \geq u\}$$

If $\beta = (\beta_1, \dots, \beta_j, \dots)$ is a vector with integral entries we define

$$V(E, \beta) = \bigcap_j V(E, j, \beta_j)$$

and

$$(4.3) \quad V(E, \geq \beta) = \bigcap_j V(E, j, \geq \beta_j)$$

We consider a monomial ideal E and its associated sequence m_0, \dots, m_t . The ideals in $V(E)$ are parametrized by the affine space \mathbf{A}^n where $n = \#\mathcal{S}(E)$. We denote by p_{ij} with $(i, j) \in \mathcal{S}(E)$ (or simply by p_1, \dots, p_n) the coordinates of \mathbf{A}^n . Given $p \in \mathbf{A}^n$ we consider the matrix $N \in T_3(E)$, defined in (3) of Definition 3.2. Set $M(p) = M_0(E) + N$. By the Hilbert-Burch theorem, the ideal I of maximal minors of $M(p)$ has the free resolution:

$$(4.4) \quad 0 \rightarrow \bigoplus_{i=1}^t R(-b_i) \xrightarrow{M(p)} \bigoplus_{i=1}^{t+1} R(-a_i) \rightarrow 0$$

where $a_i = t + 1 - i + m_{i-1}$ for $i = 1, \dots, t+1$ and $b_i = a_{i+1} + 1$ for $i = 1, \dots, t$. For every j we set

$$w_j = \{i : a_i = j\} \quad \text{and} \quad v_j = \{i : b_i = j\}.$$

Tensoring 4.4 with k and taking the degree j component we have the complex of vector spaces

$$k^{\#v_j} \xrightarrow{M(p)_j} k^{\#w_j} \rightarrow 0$$

whose homology gives the Betti numbers of I . Here $M(p)_j$ is the submatrix of $M(p)$ with rows indices w_j and column indices v_j .

It follows that

$$(4.5) \quad \beta_{0,j}(I) = \#w_j - \text{rank } M(p)_j$$

and hence $V(E, j, \geq u)$ is the determinantal variety defined by the condition

$$\text{rank } M(p)_j \leq \#w_j - u.$$

If $i_1 \in w_j$ and $i_2 \in v_j$ then (i_1, i_2) -th entry of $M(p)$ is:

$$\begin{array}{ll} p_{i_1 i_2} & \text{if } i_1 > i_2 \text{ and } d_{i_2} > 0, \\ 0 & \text{if } i_1 > i_2 \text{ and } d_{i_2} = 0, \\ 1 & \text{if } i_1 = i_2, \\ 0 & \text{if } i_1 < i_2. \end{array}$$

Hence the matrices $M(p)_j$ have entries that are either variables or 0 or 1. Furthermore the sets of the variables involved in $M(p)_j$ and in $M(p)_i$ are disjoint if $i \neq j$. To summarize:

Lemma 4.1. *The variety $V(E, \geq \beta)$ is the transversal intersection of the determinantal varieties $V(E, j, \geq \beta_j)$. In particular, the codimension of $V(E, \geq \beta)$ is the sum of the codimensions of the $V(E, j, \geq \beta_j)$ and $V(E, \geq \beta)$ is irreducible iff $V(E, j, \geq \beta_j)$ is irreducible for every j .*

From now on we concentrate our attention on the variety $V(E, j, \geq u)$. If $i \in w_j \cap v_j$ then (i, i) -entry of $M(p)_j$ is 1 and all the other entries in that column are 0. So we can simply get rid of the column and the row containing the 1's. Denote by $M(p)_j^*$ the submatrix that we get from $M(p)_j$ removing the 1's together with their columns and rows. Since the 1's are in different rows and columns we have

$$\text{rank } M(p)_j = \text{rank } M(p)_j^* + \#(w_j \cap v_j)$$

Noticing that $\#(w_j \setminus w_j \cap v_j)$ is exactly $\beta_{0,j}(E)$ we can conclude that:

Lemma 4.2. *The variety $V(E, j, \geq u)$ is defined by the condition*

$$\text{rank } M(p)_j^* \leq \beta_{0,j}(E) - u$$

The matrices $M(p)_j^*$ have entries that are either 0 or distinct variables and if the (i_1, i_2) -th entry is 0 the same is true also for the (h_1, h_2) -th with $h_1 \leq i_1$ and $h_2 \geq i_2$, that is, they look like

$$(4.6) \quad \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

where each \bullet is a distinct variable.

Remark 4.3. *The ideals of minors of a given size of the matrices of type 4.6 are radical (to prove it one can use Gröbner bases) but obviously not prime in general. They can have clearly minimal primes of different codimension.*

The following example shows that $V(E, \geq u)$ is not irreducible in general.

Example 4.4. Let $E = (x^6, x^5y, x^4y^3, x^3y^4, x^2y^4, xy^5, y^7)$. Here $d = (1, 2, 1, 0, 1, 2)$ and $a = (6, 6, 7, 7, 6, 6, 7)$ and $b = (7, 8, 8, 7, 7, 8)$. We have that $V(E)$ is an 8-dimensional affine space parametrized by the matrix

$$M(p) = \begin{array}{c|cccccc} & 7 & 8 & 8 & 7 & 7 & 8 \\ \hline 6 & y & 0 & 0 & 0 & 0 & 0 \\ 6 & -x & y^2 & 0 & 0 & 0 & 0 \\ 7 & p_1 & -x + p_4y & y & 0 & 0 & 0 \\ 7 & p_2 & p_5y & -x & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & -x & y & 0 \\ 6 & 0 & 0 & 0 & 0 & -x & y^2 \\ 7 & p_3 & p_6y & 0 & 0 & p_7 & -x + p_8y \end{array}$$

The numbers on the boundary are the degree of the syzygies (the first row) and degree of the generators (the first column).

Here the only interesting variety is $V(E, 7, \geq u)$. We have $w_7 = \{3, 4, 7\}$ and $v_7 = \{1, 4, 5\}$. The matrix $M(p)_7$ is obtained by $M(p)$ by selecting the rows and columns marked with 7:

$$M(p)_7 = \begin{array}{c|ccc} & 7 & 7 & 7 \\ \hline 7 & p_1 & 0 & 0 \\ 7 & p_2 & 1 & 0 \\ 7 & p_3 & 0 & p_7 \end{array}$$

To get $M(p)_7^*$ we have to cancel rows and columns containing 1's:

$$M(p)_7^* = \begin{pmatrix} p_1 & 0 \\ p_3 & p_7 \end{pmatrix}$$

Hence $V(E, 7, \geq u)$ is defined by the condition

$$\text{rank } M(p)_7^* \leq 2 - u$$

Therefore $V(E, 7, \geq 1)$ is defined by $p_1p_7 = 0$ and has two irreducible components of codimension 1. The variety $V(E, 7, \geq 2)$ is defined by $p_1 = p_3 = p_7 = 0$ and is irreducible of codimension 3.

The above example can be generalize to show that every matrix of type 4.6 can arise as $M(p)_j^*$ for some E and some j . Instead of given complicated and cumbersome details, we just give an example (hopefully illuminating) leaving the details to the interested readers.

Example 4.5. Starting with E associated to the sequence

$$d = (1, 1, 2, 1, 0, 1, 1, 1, 2, 1, 1, 0, 1, 1, 2, 1, 1, 1)$$

the matrix $M(p)_{19}^*$ is:

$$\begin{array}{ccccccc}
\bullet & \bullet & 0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

and $V(E, 19, \geq u)$ is defined by condition $\text{rank } M(p)_{19}^* \leq 7 - u$.

If E is a lex-segment then $d_i > 0$ for every i . This has the effect that the matrices $M(p)_j$ are matrices of indeterminates. Hence we obtain the following results of Iarrobino [I2]:

Corollary 4.6. *Let L be a lex-segment. Then variety $V(L, j, \geq u)$ is defined by the condition $\text{rank } M(p)_j \leq \beta_{0,j}(L) - u$ where $M(p)_j$ is a matrix of distinct variables of size $\beta_{0,j}(L) \times \beta_{1,j}(L)$. In particular:*

- (1) $V(L, j, \geq u)$ is irreducible. It coincides with the closure of $V(L, j, u)$ provided $V(L, j, u)$ is not empty, that is, provided $\beta_{0,j}(L) - \beta_{1,j}(L) \leq u \leq \beta_{0,j}(L)$.
- (2) If $\beta_{0,j}(L) - \beta_{1,j}(L) \leq u \leq \beta_{0,j}(L)$ then the codimension of $V(L, j, \geq u)$ is $(\beta_{1,j}(L) - \beta_{0,j}(L) + u)u$.

If I is an ideal with the same Hilbert function of the lex-segment L and $\beta_{0,j}(I) = u$ then $\beta_{1,j}(L) - \beta_{0,j}(L) + u$ is exactly $\beta_{1,j}(I)$. Hence the formula for the codimension of $V(L, j, \geq u)$ can be written as $\beta_{1,j}(I)\beta_{0,j}(I)$.

It follows that:

Corollary 4.7. *Let L be a lex-segment ideal and I a homogeneous ideal with the Hilbert function of L . Set $\beta = \{\beta_{0,j}(I)\}$. Then the variety $V(L, \geq \beta)$ is irreducible, it is the closure of $V(L, \beta)$ and it has codimension $\sum_j \beta_{1,j}(I)\beta_{0,j}(I)$.*

We conclude the paper with an example.

Example 4.8. Let $L = (x^8, x^7y, x^6y^2, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^9, y^{10})$. Then $V(L)$ is \mathbf{A}^{22} , the parametrization given via the matrix $M(p)$

	9	9	10	10	10	10	11	11
8	y	0	0	0	0	0	0	0
8	$-x$	y	0	0	0	0	0	0
8	0	$-x$	y^2	0	0	0	0	0
9	p_1	p_5	$-x + yp_9$	y	0	0	0	0
9	p_2	p_6	yp_{10}	$-x$	y	0	0	0
9	p_3	p_7	yp_{11}	0	$-x$	y	0	0
9	p_4	p_8	yp_{12}	0	0	$-x$	y^2	0
10	0	0	p_{13}	p_{15}	p_{17}	p_{19}	$-x + yp_{21}$	y
10	0	0	p_{14}	p_{16}	p_{18}	p_{20}	yp_{22}	$-x$

The matrices whose ranks describe the Betti strata are:

$$M(p)_9 = \begin{pmatrix} p_1 & p_5 \\ p_2 & p_6 \\ p_3 & p_7 \\ p_4 & p_8 \end{pmatrix} \quad \text{and} \quad M(p)_{10} = \begin{pmatrix} p_{13} & p_{15} & p_{17} & p_{19} \\ p_{14} & p_{16} & p_{18} & p_{20} \end{pmatrix}$$

For instance, with $\beta = (\beta_j)$ defined by $\beta_9 = 3, \beta_{10} = 1$ and $\beta_j = \beta_j(L)$ for $j \neq 9, 10$ the Betti strata $V(L, \geq \beta)$ is describe by $\text{rank } M(p)_9 \leq 1$ and $\text{rank } M(p)_{10} \leq 1$.

REFERENCES

- [AS] K.Altmann, B.Sturmfels, *The graph of monomial ideals*. J. Pure Appl. Algebra 201 (2005), no. 1-3, 250–263.
- [BB1] A.Bialynicki-Birula, *Some Theorems on Actions of Algebraic Groups*. Annals of Mathematics Vol. 98, No. 3 (Nov., 1973), pp. 480-497.
- [BB2] A.Bialynicki-Birula, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*. Bulletin de l'Académie Polonaise des Sciences, Série des sciences math. astr. et phys. 24(9), (1976) 667–674.
- [Br] J.Briançon, *Description de $\text{Hilb}^n \mathbf{C}\{x, y\}$* . Invent. Math. 41 (1977) 45–89.
- [BH] W.Bruns, J.Herzog, *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press 1993.
- [Co] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*. Available at <http://cocoa.dima.unige.it>
- [CRV] A.Conca, M.E.Rossi, G.Valla, *Grbner flags and Gorenstein algebras*. Compositio Math. 129 (2001), no. 1, 95–121.
- [E] D.Eisenbud, *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
- [ES1] G.Ellingsrud, S.A.Strømme, *On the homology of the Hilbert scheme of points in the plane*. Invent. Math. 87, (1987) 343–352.
- [ES2] G.Ellingsrud, S.A.Strømme, *On a cell decomposition of the Hilbert scheme of points in the plane*. Invent. Math. 91, (1988). 365–370.
- [Go1] L.Göttsche, *Betti-numbers of the Hilbert scheme of points on a smooth projective surface*. Math. Ann. 286 (1990). 193–207.
- [Go2] L.Göttsche, *Betti-numbers for the Hilbert function strata of the punctual Hilbert scheme in two variables*. Manuscripta Math. 66 (1990) 253–259.
- [Gt] G.Gotzmann, *A stratification of the Hilbert scheme of points in the projective plane*. Math. Zeitschrift 199(4) (1988). 539–547.
- [I1] A.Iarrobino, *Punctual Hilbert schemes*. Mem. Amer. Math. Soc. 10 (188) (1977).
- [I2] A.Iarrobino, *Betti strata of height two ideals*. Journal of Algebra 285 (2005). 835–855.
- [IY] A.Iarrobino, J.Yameogo, *The family G_T of graded Artinian quotients of $k[x, y]$ of given Hilbert function*, Special issue in honor of Steven L. Kleiman. Comm. Algebra 31 (2003), no. 8, 3863–3916.
- [KR] M.Kreuzer, L.Robbiano, *Computational commutative algebra 1*. Springer-Verlag, Berlin, 2000.
- [MS] E.Miller, B.Sturmfels, *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, I-16146 GENOVA, ITALY

E-mail address: conca@dimma.unige.it, valla@dimma.unige.it